On the reduced game of a cooperative game and its solution

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Abstract. We define a partially consistent property for a solution of cooperative TU games, then use the consistent property to characterize the well-known Shapley value.

Introduction. In 1989, Hart and Mas-Colell [1] were the first to introduce the potential approach to traditional UT games. In consequence, they proved that the traditional Shapley value [8] can result as the vector of marginal contributions of a potential. The potential approach is also shown to yield a characterization for the Shapley value, particularly in terms of an internal consistency property.

In 1992, Hsiao and Raghavan [2, 3] extended the traditional cooperative game to a multi-choice cooperative game and extended the traditional Shapley value to a multi-choice Shapley value. In 1994, Hsiao, Yeh and Mo [4] extended some interesting results in [1] to the multi-choice Shapley value. But, the authors in [4] got stuck with extending Hart and Mas-Colell's [1] axiomatization of the traditional Shapley value to the multi-choice Shapley value. In this article, we find out the reason why the authors were stuck with the problem.

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In this article, we find that the reduced game defined by Hart and Mas-Colell in 1989 was not well-defined. Therefore, the consistent property based on the reduced game was not well-defined either. Hence a characterization of the Shapley value proposed by Hart and Mas-Colell was incomplete. In this article, we make the reduced game well-defined in a nature way and amend the characterization of the Shapley value by applying a partially consistent property.

Definitions and Notations. Following [1] and chapter 9 in [9], we have the following definitions and notations. Let N be a finite set of players and $|N|$ denote the number of players in N.

A cooperative game with side payments - in short, *a game* - consists of a pair (N, v), where N is a finite set of players and $v : 2^N \to R$ is the *characteristic function* satisfying

$$v(\emptyset) = 0.$$

A subset $S \subset N$ is called a *coalition*.

Let $\mathbf{G}$ denote the set of all games. Formally, a solution function ϕ is a function defined on $\mathbf{G}$ that associated to every $(N, v) \in \mathbf{G}$ a payoff vector $\phi(N, v)= (\phi^i(N, v))_{i\in N} \in R^n$.

Given a solution function ϕ, a game (N, v) and a coalition $T \subset N$, the reduced game is defined by

$$v_T^\phi(S) = v(S \cup T^c) - \sum_{i\in T^c} \phi^i(S \cup T^c, v)$$

for all $S \subset T$, where $T^c = N\backslash T$. The solution function ϕ is *consistent* if

$$\phi^j(T, v_T^\phi) = \phi^j(N, v)$$

for every game (N, v), every coalition $T \subset N$ and all $j \in T$.

Remark 1. Before we recognize v_T^ϕ as a game, we have to provide that

$$v_T^\phi(\emptyset) = v(T^c) - \sum_{i\in T^c} \phi^i(T^c, v) = 0$$

That is

$$v(T^c) = \sum_{i \in T^c} \phi^i(T^c, v).$$

In other words, ϕ is efficient for (T^c, v).

But, in the beginning of the definition, we did not provide that ϕ is efficient, i.e. we did not provide the sufficient condition which makes v_T^ϕ a game. In particular, we even did not provide that

$$(***) \qquad \phi^1(\{1\}, v) = 0,$$

for the trivial one-person game $(\{1\}, v)$ where $v(\{1\}) = v(\emptyset) = 0$.

Therefore, given a two-person game $(\{1, i\}, v)$ such $v(\{1\}) = v(\emptyset) = 0$ and $v(\{1, i\}) = v(\{i\}) \neq 0$, for $T = \{i\}$ and ϕ, we can not say that the reduced game v_T^ϕ is a game before we provide (***).

Since ϕ is defined on the set of all games, if the reduced game v_T^ϕ is not a game then $\phi(v_T^\phi)$ is not defined, then the consistent property is not well-defined.

To make this article self-contained, we copy the definition of *standard for two-person games*, Theorem B and part of its proof, form page 598 and page 599 in [1], as follows.

A solution is standard for two-person games if

$$(1.1) \qquad \phi^i(\{i, j\}, v) = v(\{i\}) + \frac{1}{2}[v(\{i, j\}) - v(\{i\}) - v(\{j\})]$$

for all $i \neq j$ and all v. Thus, the "surplus " $[v(\{i, j\}) - v(\{i\}) - v(\{j\})]$ is equally divided among the two players. Most solutions satisfy this requirement, in particular, the Shapley and the nucleolus.

Theorem B. Let ϕ be a solution function. Then ϕ is (i)consistent and (ii) standard for two-person games, if only if ϕ is the Shapley value.

We now copy, from [1], the proof that if ϕ satisfies (i) and (ii) then ϕ is efficient as follows.

Proof. Assume ϕ satisfy (i) and (ii). We claim first that ϕ is efficient, i.e.,

$$\sum_{i\in N}\phi^i(N,v)=v(N) \tag{1.2}$$

for all (N,v). This indeed holds for $|N|=2$ by (1.1). Let $n\geq 3$, and assume (1.2) holds for all games with less than n players. For a game (N,v) with $|N|=n$, let $i\in N$; by consistency

$$\sum_{j\in N}\phi^j(N,v)=\sum_{j\in N\backslash\{i\}}\phi^j(N\backslash\{i\},v_{-i})+\phi^i(N,v)$$

where $v_{-i}\equiv v^{\phi}_{N\backslash\{i\}}$. By assumption, ϕ is efficient for games with $n-1$ players, thus

$$=v_{-i}(N\backslash\{i\})+\phi^i(N,v)=v(N)$$

(by definition of v_{-i}). Therefore ϕ is efficient for all $n\geq 2$.

Finally, for $|N|=1$, we have to show that $\phi^i(\{i\},v)=v(\{i\})$. Indeed, let $v(\{i\})=c$, and consider the game $(\{i,j\},\bar{v})$(for some $j\neq i$), with $\bar{v}(\{i\})=\bar{v}(\{i,j\})=c$, $\bar{v}(\{j\})=0$. By (ii), $\phi^i(\{i,j\},\bar{v})=c$ and $\phi^j(\{i,j\},\bar{v})=0$; hence $\bar{v}_{-j}(\{i\})=c-0=c=v(\{i\})$, and $c=\phi^i(\{i,j\},\bar{v})=\phi^i(\{i\},\bar{v}_{-j})=\phi^i(\{i\},v)$ by consistency. This concludes the proof of the efficiency of ϕ.

Note 1. The above proof, by Hart and Mas-Colell, of the efficiency of ϕ is incomplete, or say, has an error. Let's check the final statement of the proof:

$$c=\phi^i(\{i,j\},\bar{v})=\phi^i(\{i\},\bar{v}_{-j})=\phi^i(\{i\},v). \tag{1.3}$$

We need to prove that $\bar{v}_{-j}\equiv v$ before we claim $\phi^i(\{i\},\bar{v}_{-j})=\phi^i(\{i\},v)$, i.e. we have to prove

$$\bar{v}_{-j}(\emptyset)=0=v(\emptyset) \tag{1.4}$$

and

$$\bar{v}_{-j}(\{i\})=c-0=c=v(\{i\}). \tag{1.5}$$

Now, (1.4) holds if and only if $\bar{v}_{-j}(\emptyset) = \bar{v}^{\phi}_{\{i\}}(\emptyset) = 0$, i.e., $\bar{v}^{\phi}_{\{i\}}(\emptyset) = \bar{v}(\{j\}) - \phi^j(\{j\}, \bar{v}) = 0$. Therefore, (1.4) holds if and only if $\bar{v}(\{j\}) = \phi^j(\{j\}, \bar{v})$

That is, we have to provide that ϕ is efficient for the one-person game $(\{j\}, \bar{v})$ before we claim that (1.4) hold. Please note that no matter if j is dummy or not, ϕ is efficient for $(\{j\}, \bar{v})$ if and only if $\bar{v}(\{j\}) = \phi^j(\{j\}, \bar{v})$.

In other words, let player j in the above proof be the player 1 in (***), we find that without (***), we can not reduce the two-person game $(\{i, j\}, v)$ to one person game$(\{i\}, v^{\phi}_{\{i\}})$. Therefore, using (i) and (ii) by adding a dummy player to show that ϕ is efficient for $|N| = 1$ is incorrect.

Main Results. There is an interpretation of intuitive meaning of consistency in [1] as follows: Let ϕ be a function that associates a payoff to every player in every game. For any group T of players in a game, one defines a "reduced game" among them by giving the rest of players (in T^c) payoffs according to ϕ. Then ϕ is said to be consistent if, when it is applied to any "reduced game", it yields the same payoffs as in the original game.

A cheap way to make the reduced game well-defined is just defining the reduced game as follows.

$$v^{\phi}_T(S) = \begin{cases} v(S \cup T^c) - \sum_{i \in T^c} \phi^i(S \cup T^c, v) & \text{when } S \subseteq T \text{ and } S \neq \emptyset \\ 0 & \text{when } S = \emptyset. \end{cases}$$

But, it will be very controversial in case $v(\emptyset \cup T^c) - \sum_{i \in T^c} \phi^i(\emptyset \cup T^c, v) \neq 0$ and we define $v^{\phi}_T(\emptyset) = 0$. Therefore, we suggest the following definitions.

Definition 1. Given a solution function ϕ, a game (N, v) and a coalition $T \subset N$ and $T \neq \emptyset$ the reduced function with respect to T and ϕ is defined by

$$v^{\phi}_T(S) = v(S \cup T^c) - \sum_{i \in T^c} \phi^i(S \cup T^c, v)$$

for all $S \subseteq T$, where $T^c = N \backslash T$. Furthermore, if v^{ϕ}_T satisfies

$$v^{\phi}_T(\emptyset) = v(\emptyset \cup T^c) - \sum_{i \in T^c} \phi^i(\emptyset \cup T^c, v) = 0,$$

then we call v^{ϕ}_T a reduced game.

Definition 2. Let ϕ be a solution function defined on $\mathbf{G}$ such that for some $(N, v) \in \mathbf{G}$ and some $T \subset N$

$$\phi^j(T, v_T^\phi) = \phi^j(N, v), \tag{2.1}$$

holds for all $j \in T$ whenever the reduced function v_T^ϕ is a reduced game, then ϕ is said to be *partially consistent*.

If every reduced function v_T^ϕ is a reduced game for every game (N, v) and every coalition $T \subset N$ and (2.1) holds for all $j \in T$, then ϕ is said to be *consistent*.

Example 1. Let ϕ be a solution function defined on $\mathbf{G}$ such that $\phi^j(N, v) = 0$, for all $(N, v) \in \mathbf{G}$ and all $j \in N$, apparently the reduced function v_T^ϕ is a reduced game if and only if $v_T^\phi(\emptyset) = v(T^c) = 0$. Of course there are some $(N, v) \in \mathbf{G}$ with $v(T^c) = 0$ for some $T \subset N$ and $v(T^c) \neq 0$ for the other T's. Therefore, ϕ is partially consistent instead of consistent.

We now construct a non-trivial partially consistent solution which is not consistent as the following.

Example 2. Given $\epsilon > 0$, assign a solution function ϕ on $\mathbf{G}$ as follows:
(i) ϕ is efficient for one and only one particular game $(\{1\}, v)$ with particular player 1 where $v(\{1\}) = 1 + \epsilon$ and $v(\emptyset) = 0$, i.e., assign $\phi^1(\{1\}, v) = 1 + \epsilon$.
(ii) ϕ is not efficient for any $(N, v) \in \mathbf{G}$ except the particular game $(\{1\}, v)$ in (i), i.e.,

$$\sum_{j \in N} \phi^j(N, v) \neq v(N)$$

for any $(N, v) \in \mathbf{G}$ with $(N, v) \neq (\{1\}, v)$, the particular game in (i). This always can be done by making

$$\sum_{j \in N} \phi^j(N, v) = v(N) \pm k \cdot \epsilon$$

for some real number $k \cdot \epsilon$.
(iii) By (i)and (ii) we know that given $(N, v) \in \mathbf{G}$, the reduced function v_T^ϕ is a reduced game if and only if $T = N \backslash \{1\}$ and the particular game $(\{1\}, v)$ is a sub-game of (N, v).

If $1 \in N$, the particular game $(\{1\}, v)$ is a sub-game of (N, v) and $(N\backslash\{1\}, v^{\phi}_{N\backslash\{1\}})$ is a reduced game, we assign

$$\phi^j(N\backslash\{1\}, v^{\phi}_{N\backslash\{1\}}) = \phi^j(N, v)$$

for every $j \in N\backslash\{1\}$. In case the above assignment makes

$$\sum_{j \in N\backslash\{1\}} \phi^j(N\backslash\{1\}, v^{\phi}_{N\backslash\{1\}}) = v^{\phi}_{N\backslash\{1\}}(N\backslash\{1\}) = v(N) - \phi^1(N, v),$$

then we can always change the number $k \cdot \epsilon$ to make

$$\sum_{j \in N\backslash\{1\}} \phi^j(N\backslash\{1\}, v^{\phi}_{N\backslash\{1\}}) \neq v^{\phi}_{N\backslash\{1\}}(N\backslash\{1\}).$$

Define $v^{\phi}_{N\backslash\{1\}} \equiv \bar{v}$ then ϕ is not efficient for $(N\backslash\{1\}, \bar{v})$. This makes (iii) never contradict with (i) or (ii). Hence, ϕ is *partially consistent* instead of *consistent*.

The solution ϕ in example 2 looks artificial, but it does happen in the real world. A dictator, or say, a regime takes the right to choose ϕ to allocate payoffs among the people (players). Usually, this ϕ is inefficient and is full of discriminations. Ironically, the dictator will say that his way of allocating payoffs among the people is consistent. But, in fact, just partially consistent. However, because of cultural differences, different peoples have different reactions to different solutions. Therefore, in this article, we do not intend to evaluate the rationality of the solutions.

Note 2. A partially consistent solution is not necessary a consistent solution. However, if we are simply interested in a specified game (N, v) and all its sub-games (T, v) where $T \subset N$ and if the solution ϕ satisfies,

$$\phi^j(T, v^{\phi}_T) = \phi^j(N, v),$$

only for the specified game (N, v), every $T \subset N$ and all $j \in T$ instead of every $(N, v) \in \mathbf{G}$, then we are satisfied with ϕ although it is only partially consistent.

Definition 3. Let $\mathbf{G}_1 \subset \mathbf{G}$ be the set of all one-person games. If there exists a particular one-person game $(\{1\}, v) \in \mathbf{G}_1$ such that $v(\emptyset) = 0$ and $v(\{1\}) = k$ where k is a constant, a solution function ϕ defined on $\mathbf{G}$ is said to be *one-person-k efficient* if and only if $\phi^1(\{1\}, v) = v(\{1\}) = k$.

Definition 4. Given a game (N, v) a player i is said to be a *non-essential player* if $v(\{i\}) = k$ for some constant k and $v(S \cup \{i\}) = v(S) + v(\{i\}) = v(S) + k$ for all $S \subset N$ with $i \notin S$. If $k = 0$, we call player i a *dummy player*. *Dummy player* is a special case of *non-essential player*.

Given $(N, v) \in \mathbf{G}$ where $N = \{1, 2, ..., n\}$, allow a new player, say $(n + 1)$, to join the game, then we have a new set of players $N^* = N \cup \{n + 1\}$.

Let $\bar{v}(S) = v(S)$, for all $S \subseteq N$. Assign $\bar{v}(\{n + 1\})$ a value k not necessarily zero. Then we can define a new game $(N^*, \bar{v})$, such that $n+1$ is a non-essential player in $(N^*, \bar{v})$. We call $(N^*, \bar{v})$ a non-essential extension of (N, v). A solution ϕ of (N, v) is said to be *independent of non-essential players* if $\phi^i(N, v) = \phi^i(N^*, \bar{v})$, for all $i \in N$. Otherwise, ϕ is said to be *dependent of non-essential players*.

In case the player $n+1$ is dummy in $(N^*, \bar{v})$, then we say $(N^*, \bar{v})$ is a dummy extension of (N, v). Accordingly, ϕ is said to be *dummy free* if $\phi^i(N, v) = \phi^i(N^*, \bar{v})$, for all $i \in N$. Otherwise, ϕ is said to be *dependent of dummy players*.

Theorem 1. Let ϕ be a solution function. If ϕ is (i) one-person-k efficient for a constant k, (ii) standard for two-person games and (iii) partially consistent, then ϕ is efficient, accordingly ϕ is one-person-k efficient for all finite k.

Proof. We shall prove

$$\sum_{i \in N} \phi^i(N, v) = v(N) \tag{2.2.1}$$

for all (N, v). By (i), without loss of generality, we may assume ϕ is one-person-k efficient for $k = 1$.

Given any one-person game $(\{i\}, v) \in \mathbf{G}$, consider its non-essential extension game $(\{i, 1\}, \bar{v})$ such that $\bar{v}(\emptyset) = v(\emptyset) = 0$, $\bar{v}(\{i\}) = v(\{i\})$. Assign $\bar{v}(\{1\}) = 1$ and $\bar{v}(\{i, 1\}) = v(\{i\}) + \bar{v}(\{1\}) = v(\{i\}) + 1$, then $(\{i, 1\}, \bar{v})$ is well-defined.

Then by (ii), standard for two-person games, we have

$$\phi^i(\{i, 1\}, \bar{v}) = \bar{v}(\{i\}) = v(\{i\}) \text{ and } \phi^1(\{i, 1\}, \bar{v}) = \bar{v}(\{1\}) = 1 \tag{2.2.2}$$

Let $\{i\} = T$, consider the reduced function

$$\bar{v}^{\phi}_{\{i\}}(S) = \bar{v}(S \cup T^c) - \sum_{j \in T^c} \phi^j(S \cup T^c, \bar{v})$$

we have

(2.2.3) $\bar{v}^{\phi}_{\{i\}}(\{i\}) = \bar{v}(\{i\} \cup \{1\}) - \phi^1(\{i,1\},\bar{v}) = (\bar{v}(\{i\})+1) - 1 = v(\{i\}) + 1 - 1 = v(\{i\})$

and

$$\bar{v}^{\phi}_{\{i\}}(\emptyset) = \bar{v}(\emptyset \cup \{1\}) - \phi^1(\{1\},\bar{v}) = \bar{v}(\{1\}) - \phi^1(\{1\},\bar{v})$$

Now, for the one-person game $(\{1\},\bar{v})$ where $\bar{v}(\emptyset) = 0$ and $\bar{v}(\{1\}) = 1$, since ϕ is one-person-k efficient for $k = 1$, then $\phi^1(\{1\},\bar{v}) = 1 = \bar{v}(\{1\})$

Therefore,

$$\bar{v}^{\phi}_{\{i\}}(\emptyset) = \bar{v}(\{1\}) - \phi^1(\{1\},\bar{v}) = 1 - 1 = 0 = v(\emptyset) \tag{2.2.4}$$

By (2.2.3) and (2.2.4), we know that $\bar{v}^{\phi}_{\{i\}}$ is the reduced game with respect to $\{i\}$ and ϕ. Moreover, we get

$$(\{i\},v) \equiv (\{i\},\bar{v}^{\phi}_{\{i\}})$$

Hence,

$$\phi^i(\{i\},v) = \phi^i(\{i\},\bar{v}^{\phi}_{\{i\}}) \tag{2.2.5}$$

Next, since $\bar{v}^{\phi}_{\{i\}}$ is a reduced game and ϕ is partially consistent, then by (2.2.2) we have

$$\phi^i(\{i\},\bar{v}^{\phi}_{\{i\}}) = \phi^i(\{i,1\},\bar{v}) = \bar{v}(\{i\}) = v(\{i\}).$$

Hence, by (2.2.5), we obtain

$$\phi^i(\{i\},v) = v(\{i\}).$$

Therefore (2.2.1) holds for all $|N| = 1$ and any reduced function v^{ϕ}_T with $|T^c| = 1$ is a well-defined reduced game.

As a matter of fact (2.2.1) holds for $|N| = 2$ by (ii), therefore, any reduced function v^{ϕ}_T with $|T^c| = 2$ is a well-defined reduced game. Let $n \geq 3$, and assume (2.2.1) holds for all games with less than n players. For a game (N,v) with $|N| = n$, let $i \in N$; since $v^{\phi}_{N\backslash\{i\}}$ is a reduced game, then by partially consistency, we have

$$\sum_{j\in N} \phi^j(N,v) = \sum_{j\in N\backslash\{i\}} \phi^j(N\backslash\{i\},v^{\phi}_{N\backslash\{i\}}) + \phi^i(N,v)$$

By assumption, ϕ is efficient for games with $n-1$ players, thus

$$\sum_{j \in N} \phi^j(N,v) = v^{\phi}_{N\backslash\{i\}}(N\backslash\{i\}) + \phi^i(N,v) = v(N).$$

Therefore ϕ is efficient for all $n \geq 2$, and the proof is complete. $\diamondsuit$

Corollary 1. Let ϕ be a solution function. If ϕ is (i) one-person-k efficient for a constant k, (ii) standard for two-person games and (iii) partially consistent, then ϕ consistent.

Proof. Following the proof of Theorem 1, by mathematical induction on $|T^c|$, we can show that every reduced function (T, v^{ϕ}_T) is a reduced game for every game (N,v) and every coalition $T \subset N$. Then by (iii) ϕ is consistent. $\diamondsuit$

Remark 2. Since 1989, readers of [1] thought that if ϕ is consistent and standard for two-person games, then by just adding a dummy player, they can easily show that ϕ is efficient for every one-person game. But that is incorrect. However, we can amend the problem easily by providing the efficiency of ϕ for just one particular one-person game rather than every one-person game.

Let the particular chosen player 1 be a dummy player in the one-person game $(\{1\}, v)$ with $v(\emptyset) = 0 = v(\{1\})$. It is very nature to assign $\phi^1(\{1\}, v) = 0$. Equivalently it is very nature to assign $\phi^1(\{1\}, v) = k$ for the one-person game $(\{1\}, v)$ with $v(\emptyset) = 0$ and $v(\{1\}) = k$.

From Theorem 1 and Corollary 1, we can see the following: In order to make the reduced games well-defined in [1] Theorem B, we need ϕ to be efficient for just a particular one-person game in $\mathbf{G}_1$ instead of every one-person game. Therefore, we may relax the consistent property to the partially consistent property and amend Theorem B in [1] as follows.

Theorem B^{***}. Let ϕ be a solution function. Then ϕ is (i) one-person-k efficient for a constant k (ii) standard for two-person games (iii) partially consistent, if only if ϕ is the Shapley value.

Now, by Theorem 1 and Theorem B^{***}, we can easily see the following Corollary.

Corollary 2. The Shapley value is independent of non-essential players, in particular, dummy free.

Proof. Let φ denote the Shapley value, by Theorem B^{***}, φ is efficient and consistent. Given any (N, v) and its non-essential extension $(N^*, \bar{v})$. Suppose $N = \{1, ..., n\}$, $N^* = N \cup \{n+1\}$ and $n+1$ be the non-essential player with $\bar{v}(\{n+1)\} = k$. Then, since φ is efficient for $(\{n+1\}, \bar{v})$ the reduced function $\bar{v}^{\varphi}_{N^* \setminus \{n+1\}}$ is a reduced game. Then by consistency, φ is independent of non-essential players. $\diamondsuit$

Furthermore, we may amend Theorem 5.7 in page 605 in [1] for the w-weighted Shapley in a similar way.

Conclusion. The main contribution of this article is the following. We make the reduced games well-defined and define a partially consistent property for a solution of cooperative TU games, then use the consistent property to characterize the well-known Shapley value.

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